Time periodic solutions of compressible fluid models of Korteweg type

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Abstract

This paper is concerned with the existence, uniqueness and time-asymptotic stability of time periodic solutions to the compressible Navier-Stokes-Korteweg system effected by a time periodic external force in $\mathbb{R}^n$. Our analysis is based on a combination of the energy method and the time decay estimates of solutions to the linearized system.

Keywords Navier-Stokes-Korteweg system; Capillary fluids; Time periodic solution; Energy estimates;

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1 Introduction

The compressible Navier-Stokes-Korteweg system for the density $\rho > 0$ and velocity $u = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$ is written as :

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \bigotimes u) + \nabla P(\rho) - \mu \Delta u - (\nu + \mu)\nabla(\nabla \cdot u) = \kappa \rho \nabla \Delta \rho + \rho f(t, x). \end{cases} \tag{1.1}$$

Here, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$, $P = P(\rho)$ is the pressure, μ, ν are the viscosity coefficients, κ is the capillary coefficient, and $f(t, x) = (f_1, f_2, f_3)(t, x)$ is a given external force. System (1.1) can be used to describe the motion of the compressible isothermal fluids with capillarity effect of materials, see the pioneering work by Dunn and Serrin [1], and also [2, 3, 4].

In this paper, we consider the problem (1.1) for (ρ, u) around a constant state $(\rho_\infty, 0)$ for $n \geq 5$, where ρ_∞ is a positive constant. Throughout this paper, we make the following basic assumptions:

(H1): μ, ν and κ are positive constants and satisfying $\nu + \frac{2}{n}\mu \geq 0$.

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(H2): $P(\rho)$ is smooth in a neighborhood of ρ_∞ satisfying $P'(\rho_\infty) > 0$.
(H3): f is time periodic with period $T > 0$.

The main purpose of this paper is to show that the problem (1.1) admits a time periodic solution around the constant state $(\rho_\infty, 0)$ which has the same period as f. By combining the energy method and the optimal decay estimates of solutions to the linearized system, we prove the existence of a time periodic solution in some suitable function space. Notice that some similar results have been obtained for the compressible Navier-Stokes equations and Boltzmann equation, cf. [8, 9, 10, 11].

Precisely, Let $N \geq n + 2$ be a positive integer, define the solution space by

$$X_M(0,T) = \left\{ (\rho, u)(t,x) \left| \begin{array}{c} \rho(t,x) \in C(0,T; H^N(\mathbb{R}^n)) \cap C^1(0,T; H^{N-2}(\mathbb{R}^n)), \\ u(t,x) \in C(0,T; H^{N-1}(\mathbb{R}^n)) \cap C^1(0,T; H^{N-3}(\mathbb{R}^n)), \\ \nabla\rho(t,x) \in L^2(0,T; H^{N+1}(\mathbb{R}^n)), \\ \nabla u(t,x) \in L^2(0,T; H^N(\mathbb{R}^n)), |||(\rho,u)||| \leq M, \end{array} \right. \right\} \tag{1.2}$$

for some positive constant M and with the norm

$$|||(\rho,u)|||^2 = \sup_{0 \leq t \leq T} \left\{ \|\rho(t)\|_N^2 + \|u(t)\|_{N-1}^2 \right\} + \int_0^T \left(\|\nabla\rho(t)\|_{N+1}^2 + \|\nabla u(t)\|_N^2 \right) dt. \tag{1.3}$$

Then the existence of the time periodic solution can be stated as follows.

Theorem 1.1. *Let $n \geq 5, N \geq n+2$. Assume the assumptions (H1)-(H3) hold, and $f(t,x) \in C(0,T; H^{N-1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$. Then there exists a small constant $\delta_0 > 0$ and a constant $M_0 > 0$ which are dependent on ρ_∞, such that if*

$$\sup_{0 \leq t \leq T} \|f(t)\|_{H^{N-1} \cap L^1} \leq \delta_0, \tag{1.4}$$

then the problem (1.1) admits a time periodic solution (ρ^{per}, u^{per}) with period T, satisfying

$$(\rho^{per} - \rho_\infty, u^{per}) \in X_{M_0}(0,T)$$

Furthermore the periodic solution is unique in the following sense: if there is another time periodic solution $(\rho_1^{per}, u_1^{per})$ satisfying (1.1) with the same f, and $(\rho_1^{per} - \rho_\infty, u_1^{per}) \in X_{M_0}(0,T)$, then $(\rho_1^{per}, u_1^{per}) = (\rho^{per}, u^{per})$.

To study the stability of the time periodic solution (ρ^{per}, u^{per}) obtained in Theorem 1.1, we consider the problem (1.1) with the following initial date

$$(\rho, u)(t,x)|_{t=0} = (\rho_0, u_0)(x) \to (\rho_\infty, 0), \quad as\ |x| \to \infty. \tag{1.5}$$

Here $\rho_0(x)$ and $u_0(x)$ is a small perturbation of the time periodic solution (ρ^{per}, u^{per}). And we have the following stability result.

Theorem 1.2. *Under the assumptions of Theorem 1.1, let (ρ^{per}, u^{per}) be the time periodic solution thus obtained. If the initial date (ρ_0, u_0) be such that $\|(\rho_0 - \rho^{per}(0), u_0 - u^{per}(0)\|_{N-1}$ is sufficiently small, then the Cauchy problem (1.1), (1.5) has a unique classical solution (ρ, u) globally in time, which satisfies*

$$\begin{aligned} &\rho - \rho^{per} \in C(0,\infty; H^{N-1}(\mathbb{R}^n)) \cap C^1(0,\infty; H^{N-3}(\mathbb{R}^n)), \\ &u - u^{per} \in C(0,\infty; H^{N-2}(\mathbb{R}^n)) \cap C^1(0,\infty; H^{N-4}(\mathbb{R}^n)). \end{aligned} \tag{1.6}$$

Moreover, there exists a constant $C_0 > 0$ such that

$$\begin{aligned} &\|(\rho-\rho^{per})(t)\|_{N-1}^2+\|(u-u^{per})(t)\|_{N-2}^2+\int_0^t\left(\|\nabla(\rho-\rho^{per})(\tau)\|_{N-1}^2+\|\nabla(u-u^{per})(\tau)\|_{N-2}^2\right)d\tau \\ &\le C_0\left(\|\rho_0-\rho^{per}(0)\|_{N-1}^2+\|u_0-u^{per}(0)\|_{N-2}^2\right), \end{aligned} \tag{1.7}$$

for any $t \geq 0$ and

$$\|(\rho-\rho^{per}, u-u^{per})\|_{L^\infty} \to 0 \ \text{as} \ t \to \infty. \tag{1.8}$$

Now we outline the main ingredients used in proving of our main results. For the proof of Theorem 1.1, thanks to the time decay estimates of solutions to the linear system (2.7) (see Lemma 2.1 below), we can show the integral in (4.5) is convergent. Based on this and the elaborate energy estimates given in Section 3, we prove the existence of time periodic solution by the contraction mapping principle. Here, similar to the case of compressible Navier-Stokes equations, Theorem 1.1 is obtained only in the case $n \geq 5$ because of the convergence of the integral in (4.5). Thus, how to deal with the case $n < 5$, especially, the physical case $n = 3$, is still an open problem. Theorem 1.2 is established by the energy method. The key ingredient in the proof of Theorem 1.2, among other things, is to get the a priori estimates, which can be done similarly to the estimates in Section 3.

There have been a lot of studies on the mathematical theory of the compressible Navier-Stokes-Korteweg system. For example, Hattori and Li [12, 13] proved the local existence and the global existence of smooth solutions in Sobolev space. Danchin and Desjardins [7] studied the existence of suitably smooth solutions in critical Besov space. Bresch, Desjardins and Lin [5] considered the global existence of weak solution, then Haspot improved their results in [6]. The local existence of strong solutions was proven in [14]. Recently, Wang and Tan [15] established the optimal decay rates of global smooth solutions without external force. Li [16] discussed the global existence and optimal L^2-decay rate of smooth solutions with potential external force.

The rest of the paper is organized as follows. In Section 2, we will reformulate the problem and give some preliminaries for later use. In Section 3, we give the energy estimates on the linearized system (2.4). The proof of Theorem 1.1 is given in Section 4. In the last section, we will study the stability of the time periodic solution.

Notations: Throughout this paper, for simplicity, we will omit the variables t, x of functions if it does not cauchy any confusion. C denotes a generic positive constant which may vary in different estimates. $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^n)$. The norm in the usual Sobolev Space $H^s(\mathbb{R}^n)$ are denoted by $\|\cdot\|_s$ for $s \geq 0$. When s=0, we will simply use $\|\cdot\|$. Moreover, we denote $\|\cdot\|_{H^s} + \|\cdot\|_{L^1}$ by $\|\cdot\|_{H^s\cap L^1}$. If $g = (g_1, g_2, \cdots, g_n)$, then $\|g\| = \sum_{k=1}^{n}(\|g_k\|^2)^{\frac{1}{2}}$. $\nabla = (\partial_1, \partial_2, \cdots, \partial_n)$ with $\partial_i = \partial_{x_i}, i = 1, 2, \cdots, n$ and for any integer $l \geq 0$, $\nabla^l g$ denotes all x derivatives of order l of the function g. Finally, for multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, it is standard that

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{i=1}^{n}\alpha_i.$$

2 Reformulated system and preliminaries

We reformulate the system (1.1) in this section. Firstly, set

$$\gamma = \sqrt{P'(\rho_\infty)}, \quad \kappa' = \frac{\rho_\infty}{\gamma}\kappa, \quad \mu' = \frac{\mu}{\rho_\infty}, \quad \nu' = \frac{\nu+\mu}{\rho_\infty}, \quad \lambda_1 = \frac{\gamma}{\rho_\infty}, \quad \lambda_2 = \frac{\rho_\infty}{\gamma},$$

and define the new variables

$$\sigma = \rho - \rho_\infty, \quad v = \lambda_2 u,$$

then the system (1.1) is reformulated as

$$\begin{cases} \sigma_t + \gamma \nabla \cdot v = G_1(\sigma, v), \\ v_t - \mu' \Delta v - \nu' \nabla(\nabla \cdot v) + \gamma \nabla \sigma - \kappa' \nabla \Delta \sigma = G_2(\sigma, v) + \lambda_2 f, \end{cases} \tag{2.1}$$

where

$$G_1(\sigma, v) = -\lambda_1 \nabla \cdot (\sigma v),$$
$$G_2(\sigma, v) = -\frac{\sigma}{\rho_\infty(\sigma + \rho_\infty)}(\mu \Delta v + \nu \nabla(\nabla \cdot v)) - \lambda_1 (v \cdot \nabla) v - \lambda_2 \left[\frac{P'(\sigma + \rho_\infty)}{\sigma + \rho_\infty} - \frac{P'(\rho_\infty)}{\rho_\infty} \right] \nabla \sigma.$$

Notice that G_1 and G_2 have the following properties:

$$\begin{aligned} G_1(\sigma, v) &\sim \nabla \sigma \cdot v + \sigma \nabla \cdot v, \\ G_2(\sigma, v) &\sim \sigma \Delta v + \sigma \nabla(\nabla \cdot v) + (v \cdot \nabla) v + \sigma \nabla \sigma. \end{aligned} \tag{2.2}$$

Here $\sim$ means that two side are of same order.

Set $U = (\sigma, v)$, $G = (G_1, G_2)$, $F = (0, \lambda_2 f)$ and

$$\mathbb{A} = \begin{pmatrix} 0 & \gamma div \\ \gamma \nabla - \kappa' \nabla \Delta & -\mu' \Delta - \nu' \nabla div \end{pmatrix},$$

then the system (2.1) takes the form

$$U_t + \mathbb{A} U = G(U) + F. \tag{2.3}$$

We first consider the linearized system of (2.1):

$$\begin{cases} \sigma_t + \gamma \nabla \cdot v = G_1(\tilde{U}), \\ v_t - \mu' \Delta v - \nu' \nabla(\nabla \cdot v) + \gamma \nabla \sigma - \kappa' \nabla \Delta \sigma = G_2(\tilde{U}) + \lambda_2 f, \end{cases} \tag{2.4}$$

for any given functions $\tilde{U} = (\tilde{\sigma}, \tilde{v})$ satisfying

$$\tilde{\sigma} \in H^{N+2}(\mathbb{R}^n), \quad \tilde{v} \in H^{N+1}(\mathbb{R}^n).$$

Notice that the system (2.4) can be written as

$$U_t + \mathbb{A} U = G(\tilde{U}) + F. \tag{2.5}$$

By the Duhamel's principle, the solution to the system (2.4) can be written in the mild form as

$$U(t) = \mathbb{S}(t, s) U(s) + \int_s^t \mathbb{S}(t, \tau)(G(\tilde{U}) + F)(\tau) d\tau, \quad t \geq s, \tag{2.6}$$

where $\mathbb{S}(t, s)$ is the corresponding linearized solution operator defined by

$$\mathbb{S}(t, s) = e^{(t-s)\mathbb{A}}, \quad t \geq s.$$

Indeed, the corresponding homogeneous linear system to (2.4) is

$$\begin{cases} \sigma_t + \gamma\nabla\cdot v = 0, \\ v_t - \mu'\Delta v - \nu'\nabla(\nabla\cdot v) + \gamma\nabla\sigma - \kappa'\nabla\Delta\sigma = 0, \\ \sigma|_{t=s} = \sigma_s(x), \quad v|_{t=s} = v_s(x). \end{cases} \tag{2.7}$$

By repeating the argument in the proof of Theorem 1.3 in [15], we can get the following result for the problem (2.7). The details are omitted here.

Lemma 2.1. *Let $l \geq 0$ be an integer. Assume that (σ, v) is the solution of the problem (2.7) with the initial date $\sigma_s \in H^{l+1}\cap L^1$ and $v_s \in H^l \cap L^1$, then*

$$\|\sigma(t)\| \leq C(1+t)^{-\frac{n}{4}}\left(\|(\sigma_s, v_s)\|_{L^1} + \|(\sigma_s, v_s)\|\right),$$

$$\|\nabla^{k+1}\sigma(t)\| \leq C(1+t)^{-\frac{n}{4}-\frac{k+1}{2}}\left(\|(\sigma_s, v_s)\|_{L^1} + \|(\nabla^{k+1}\sigma_s, \nabla^k v_s)\|\right),$$

$$\|\nabla^k v(t)\| \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\left(\|(\sigma_s, v_s)\|_{L^1} + \|(\nabla^{k+1}\sigma_s, \nabla^k v_s)\|\right),$$

where k is an integer satisfying $0 \leq k \leq l$.

3 Energy estimates

In this section, we will perform some energy estimates on solutions (σ, v) to problem (2.4). Throughout of this section, we assume that $f(t,x) \in H^{N-1}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ for all $t \geq 0$. For later use, we list some standard inequalities as follows. cf. [8].

Lemma 3.1. *Let m be a positive integer and $u \in H^{[\frac{n}{2}]+1}(\mathbb{R}^n)$, then*

$$\|u\|_{L^\infty}^2 \leq C\|\nabla^{m+1}u\|\|\nabla^{m-1}u\| \quad for\ n = 2m,$$

$$\|u\|_{L^\infty}^2 \leq C\|\nabla^{m+1}u\|\|\nabla^{m}u\| \quad for\ n = 2m+1.$$

Lemma 3.2. *Let m be the integer defined in Lemma 3.1 and $f, g, h \in H^{[\frac{n}{2}]+1}(\mathbb{R}^n)$, then we have*

$$(i)\left|\int_{\mathbb{R}^n} f\cdot g\cdot h\,dx\right| \leq \epsilon\|\nabla^{m-1}f\|_2^2 + C_\epsilon\|g\|^2\|h\|^2,$$

$$(ii)\left|\int_{\mathbb{R}^n} f\cdot g\cdot h\,dx\right| \leq \epsilon\|f\|_2^2 + C_\epsilon\|\nabla^{m-1}g\|_2^2\|h\|^2,$$

for any $\epsilon > 0$. Here and hereafter, C_ϵ denotes a positive constant depending only on ϵ.

We first give the energy estimate on the low order derivatives of (σ, v).

Lemma 3.3. *Let $n \geq 5$, $N \geq n+2$, then there exists two suitably small constants $d_0 > 0$ and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, it holds*

$$\begin{aligned} &\frac{d}{dt}\left(\|U(t)\|^2 + \|\nabla\sigma(t)\|^2 + d_0\langle v, \nabla\sigma\rangle(t)\right) + \|\nabla v(t)\|^2 + \|\nabla\sigma(t)\|_1^2 \\ &\leq \epsilon C\left(\|\nabla^3\sigma(t)\|_{m-2}^2 + \|\nabla^2 v(t)\|_{m-1}^2\right) + C_\epsilon C\left(\|\tilde{U}(t)\|_{m+1}^2\|\nabla\tilde{U}(t)\|_1^2 + \|f(t)\|_{L^1\cap L^2}^2\right), \end{aligned} \tag{3.1}$$

where m is defined in Lemma 3.1 and C depends only on ρ_∞, μ, ν and κ.

Proof. Multiplying $(2.4)_1$ and $(2.4)_2$ by σ and v, respectively, and integrating them over $\mathbb{R}^n$, we have from integrating by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|U\|^2+\mu'\|\nabla v\|^2+\nu'\|\nabla\cdot v\|^2\\&=\langle G_1(\tilde{U}),\sigma\rangle+\langle G_2(\tilde{U}),v\rangle+\kappa'\langle\nabla\Delta\sigma,v\rangle+\lambda_2\langle f,v\rangle\\&=I_0+I_1+I_2+I_3.\end{aligned}\tag{3.2}$$

From (2.2) and Lemma 3.2, we have

$$\begin{aligned}I_0&\leq\epsilon\|\nabla^{m-1}\sigma\|_2^2+C_\epsilon C\left(\|\nabla\tilde{\sigma}\|^2\|\tilde{v}\|^2+\|\tilde{\sigma}\|^2\|\nabla\tilde{v}\|^2\right)\\&\leq\epsilon\|\nabla^{m-1}\sigma\|_2^2+C_\epsilon C\|\tilde{U}\|^2\|\nabla\tilde{U}\|^2,\end{aligned}\tag{3.3}$$

and

$$I_1\leq\epsilon\|\nabla^{m-1}v\|_2^2+C_\epsilon C\|\tilde{U}\|^2\|\nabla\tilde{U}\|_1^2.\tag{3.4}$$

For I_2, integrating by parts and using $(2.4)_1$, (2.2) and Lemma 3.2, we deduce that

$$\begin{aligned}I_2&=-\kappa'\langle\Delta\sigma,\nabla\cdot v\rangle=\frac{\kappa'}{\gamma}\langle\Delta\sigma,\sigma_t-G_1(\tilde{U})\rangle\\&=-\frac{\kappa'}{2\gamma}\frac{d}{dt}\|\nabla\sigma\|^2-\frac{\kappa'}{\gamma}\langle\Delta\sigma,G_1(\tilde{U})\rangle\\&\leq-\frac{\kappa'}{2\gamma}\frac{d}{dt}\|\nabla\sigma\|^2+\epsilon\|\nabla^2\sigma\|^2+C_\epsilon C\|\nabla\tilde{U}\|^2\|\nabla^{m-1}\tilde{U}\|_2^2.\end{aligned}\tag{3.5}$$

For I_3, Lemma 3.1 gives

$$I_3\leq\epsilon\|\nabla^{m-1}v\|_2^2+C_\epsilon C\|f\|_{L^1}^2.\tag{3.6}$$

Since $n\geq5$, $N\geq n+2$, we have $m-1\geq1$. Substituting (3.3)-(3.6) into (3.2) yields

$$\begin{aligned}&\frac{d}{dt}\left(\|U\|^2+\|\nabla\sigma\|^2\right)+\|\nabla v\|^2+\|\nabla\cdot v\|^2\\&\leq\epsilon C\left(\|\nabla^{m-1}\sigma\|_2^2+\|\nabla^2\sigma\|^2\right)+\epsilon C\|\nabla^2 v\|_{m-1}^2+C_\epsilon C\left(\|\tilde{U}\|_{m+1}^2\|\nabla\tilde{U}\|_1^2+\|f\|_{L^1}^2\right),\end{aligned}\tag{3.7}$$

provided that ϵ is small enough, where C depends only on ρ_∞,μ,ν and κ.

Next, we estimate $\|\nabla\sigma\|^2$. Taking the L^2 inner product with $\nabla\sigma$ on both side of $(2.4)_2$ and then integrating by parts, we have

$$\begin{aligned}&\gamma\|\nabla\sigma\|^2+\kappa'\|\nabla^2\sigma\|^2\\&=-\langle v_t,\nabla\sigma\rangle+\mu'\langle\Delta v,\nabla\sigma\rangle+\nu'\langle\nabla(\nabla\cdot v),\nabla\sigma\rangle+\langle G_2(\tilde{U})+\lambda_2 f,\nabla\sigma\rangle\\&=I_4+I_5+I_6+I_7.\end{aligned}\tag{3.8}$$

Similar to (3.5), the term I_4 can be controlled by

$$\begin{aligned}I_4&=-\frac{d}{dt}\langle v,\nabla\sigma\rangle-\langle\nabla\cdot v,\sigma_t\rangle\\&=-\frac{d}{dt}\langle v,\nabla\sigma\rangle-\langle\nabla\cdot v,-\gamma\nabla\cdot v+G_1(\tilde{U})\rangle\\&\leq-\frac{d}{dt}\langle v,\nabla\sigma\rangle+2\gamma\|\nabla\cdot v\|^2+C\|\nabla^{m-1}\tilde{U}\|_2^2\|\nabla\tilde{U}\|^2.\end{aligned}\tag{3.9}$$

Integrating by parts and using the Cauchy-Schwartz inequality, it is easy to get

$$I_5 + I_6 \leq \frac{\kappa'}{4}\|\nabla^2\sigma\|^2 + C(\|\nabla v\|^2 + \|\nabla\cdot v\|^2). \tag{3.10}$$

Finally, (2.2) and the Cauchy-Schwartz inequality imply that

$$I_7 \leq \frac{\gamma}{2}\|\nabla\sigma\|^2 + C\left(\|\nabla^{m-1}\tilde{U}\|_2^2\|\nabla\tilde{U}\|_1^2 + \|f\|^2\right). \tag{3.11}$$

Combining (3.8)-(3.11), we obtain

$$\begin{aligned}&\frac{d}{dt}\langle v, \nabla\sigma\rangle + \|\nabla\sigma\|^2 + \|\nabla^2\sigma\|^2\\ &\leq C(\|\nabla v\|^2 + \|\nabla\cdot v\|^2) + C\left(\|\nabla^{m-1}\tilde{U}\|_2^2\|\nabla\tilde{U}\|_1^2 + \|f\|^2\right).\end{aligned} \tag{3.12}$$

where the constant C depends only on ρ_∞, μ, ν and κ. Multiplying (3.12) with a small constant $d_0 > 0$ and then adding the resultant equation to (3.7), one can get (3.1) immediately by the smallness of d_0 and ϵ. This completes the proof of Lemma 3.3.

Next, we derive the energy estimate on the high order derivatives of (σ, v). We establish the following lemma.

Lemma 3.4. *Let $n \geq 5$, $N \geq n+2$, then there exists two suitably small constants $d_1 > 0$ and $\epsilon_1 > 0$ such that for $0 < \epsilon \leq \epsilon_1$, it holds*

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\sigma(t)\|_N^2 + \|\nabla v(t)\|_{N-1}^2 + d_1\sum_{|\alpha|=1}^{N}\langle\partial_x^\alpha v, \partial_x^\alpha\nabla\sigma\rangle(t)\right) + \|\nabla^2\sigma(t)\|_N^2 + \|\nabla^2 v(t)\|_{N-1}^2\\ &\leq \epsilon C\|\nabla\sigma(t)\|^2 + C_\epsilon C\left(\|\nabla\tilde{U}(t)\|_{N-2}^2\|\nabla\tilde{U}(t)\|_N^2 + \|f(t)\|_{N-1}^2\right),\end{aligned} \tag{3.13}$$

where C is depending only on ρ_∞, μ, ν and κ.

Proof. For each multi-index α with $1 \leq |\alpha| \leq N$, applying ∂_x^α to $(2.4)_1$ and $(2.4)_2$ and then taking the L^2 inner product with $\partial_x^\alpha\sigma$ and $\partial_x^\alpha v$ on the two resultant equations respectively, we have from integrating by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\partial_x^\alpha\sigma\|^2 + \|\partial_x^\alpha v\|^2\right) + \mu'\|\partial_x^\alpha\nabla v\|^2 + \nu'\|\partial_x^\alpha\nabla\cdot v\|^2\\ &= \langle\partial_x^\alpha G_1(\tilde{U}), \partial_x^\alpha\sigma\rangle + \langle\partial_x^\alpha G_2(\tilde{U}), \partial_x^\alpha v\rangle + \kappa'\langle\partial_x^\alpha\nabla\Delta\sigma, \partial_x^\alpha v\rangle + \lambda_2\langle\partial_x^\alpha f, \partial_x^\alpha v\rangle\\ &= I_8 + I_9 + I_{10} + I_{11}.\end{aligned} \tag{3.14}$$

Now, we estimate I_8-I_{11} term by term. For I_8, we deduce from (2.2) and the Cauchy-Schwartz inequality that

$$\begin{aligned}I_8 &\leq \epsilon\|\partial_x^\alpha\sigma\|^2 + C_\epsilon\|\partial_x^\alpha G_1(\tilde{U})\|^2\\ &\leq \epsilon\|\partial_x^\alpha\sigma\|^2 + C_\epsilon C\left(\|\partial_x^\alpha(\nabla\tilde{\sigma}\cdot\tilde{v})\|^2 + \|\partial_x^\alpha(\tilde{\sigma}\nabla\cdot\tilde{v})\|^2\right).\end{aligned} \tag{3.15}$$

By Leibniz's formula and Minkowski's inequality, we get

$$\begin{aligned}\|\partial_x^\alpha(\nabla\tilde{\sigma}\cdot\tilde{v})\|^2 \leq{}& C(\|(\partial_x^\alpha\nabla\tilde{\sigma})\cdot\tilde{v}\|^2 + \|\nabla\tilde{\sigma}\cdot\partial_x^\alpha\tilde{v}\|^2) + C\sum_{0<|\beta|=|\alpha|-1} C_\beta^\alpha\|\partial_x^\beta\nabla\tilde{\sigma}\cdot\partial_x^{\alpha-\beta}\tilde{v}\|^2\\ &+C\sum_{0<|\beta|\leq|\alpha|-2,\, |\alpha-\beta|\leq\frac{N}{2}} C_\beta^\alpha\|\partial_x^\beta\nabla\tilde{\sigma}\cdot\partial_x^{\alpha-\beta}\tilde{v}\|^2\\ &+C\sum_{0<|\beta|\leq|\alpha|-2,\, |\alpha-\beta|>\frac{N}{2}} C_\beta^\alpha\|\partial_x^\beta\nabla\tilde{\sigma}\cdot\partial_x^{\alpha-\beta}\tilde{v}\|^2\\ ={}& J_0 + J_1 + J_2 + J_3.\end{aligned} \tag{3.16}$$

Here C_β^α denotes the binomial coefficients corresponding to multi-indices. For J_0, lemma 3.1 gives

$$\begin{aligned} J_0 &\leq C\left(\|\tilde{v}\|_{L^\infty}^2\|\partial_x^\alpha\nabla\tilde{\sigma}\|^2+\|\nabla\tilde{\sigma}\|_{L^\infty}^2\|\partial_x^\alpha\tilde{v}\|^2\right)\\ &\leq C\left(\|\nabla\tilde{v}\|_{N-5}^2\|\nabla^2\tilde{\sigma}\|_{N-1}^2+\|\nabla^2\tilde{\sigma}\|_{N-5}^2\|\nabla\tilde{v}\|_{N-1}^2\right), \end{aligned} \tag{3.17}$$

where, in the last inequality of (3.17), we have used the fact that $m-1\geq 1$ and $m+1\leq N-4$ due to $N\geq n+2$ and $n\geq 5$. Similarly, it holds that

$$J_1 \leq C\sum_{0<|\beta|=|\alpha|-1}\|\partial_x^{\alpha-\beta}\tilde{v}\|_{L^\infty}^2\|\partial_x^\beta\nabla\tilde{\sigma}\|^2\leq C\|\nabla^2\tilde{v}\|_{N-5}^2\|\nabla^2\tilde{\sigma}\|_{N-2}^2. \tag{3.18}$$

For the terms J_2 and J_3, notice that for any $\beta\leq\alpha$ with $|\alpha-\beta|\leq\frac{N}{2}$,

$$|\alpha-\beta|+m+1\leq\frac{N}{2}+\frac{n}{2}+1\leq\frac{N}{2}+\frac{N}{2}=N,$$

and for any $\beta\leq\alpha$ with $|\alpha-\beta|>\frac{N}{2}$,

$$|\beta|+m+2=|\alpha|-|\alpha-\beta|+m+2<N-\frac{N}{2}+\frac{n}{2}+2\leq N+1.$$

which implies $|\beta|+m+2\leq N$ since $|\beta|$ and m are positive integers. Hence, we deduce from Lemma 3.1 that

$$J_2 \leq C\sum_{0<|\beta|\leq|\alpha|-2,\,|\alpha-\beta|\leq\frac{N}{2}}\|\partial_x^{\alpha-\beta}\tilde{v}\|_{L^\infty}^2\|\partial_x^\beta\nabla\tilde{\sigma}\|^2\leq C\|\nabla^2\tilde{v}\|_{N-2}^2\|\nabla^2\tilde{\sigma}\|_{N-3}^2, \tag{3.19}$$

and

$$J_3 \leq C\sum_{0<|\beta|\leq|\alpha|-2,\,|\alpha-\beta|>\frac{N}{2}}\|\partial_x^\beta\nabla\tilde{\sigma}\|_{L^\infty}^2\|\partial_x^{\alpha-\beta}\tilde{v}\|^2\leq C\|\nabla^2\tilde{v}\|_{N-3}^2\|\nabla^2\tilde{\sigma}\|_{N-2}^2. \tag{3.20}$$

Putting (3.17)-(3.20) into (3.16), we arrive at

$$\|\partial_x^\alpha(\nabla\tilde{\sigma}\cdot\tilde{v})\|^2\leq C\left(\|\nabla\tilde{v}\|_{N-5}^2\|\nabla^2\tilde{\sigma}\|_{N-1}^2+\|\nabla^2\tilde{U}\|_{N-3}^2\|\nabla\tilde{U}\|_{N-1}^2\right). \tag{3.21}$$

Similarly, it holds

$$\|\partial_x^\alpha(\tilde{\sigma}\nabla\cdot\tilde{v})\|^2\leq C\left(\|\nabla\tilde{\sigma}\|_{N-5}^2\|\nabla^2\tilde{v}\|_{N-1}^2+\|\nabla^2\tilde{U}\|_{N-3}^2\|\nabla\tilde{U}\|_{N-1}^2\right). \tag{3.22}$$

Combining (3.15), (3.21) and (3.22) yields

$$I_8\leq\epsilon\|\partial_x^\alpha\sigma\|^2+C_\epsilon C\|\nabla\tilde{U}\|_{N-2}^2\|\nabla\tilde{U}\|_N^2. \tag{3.23}$$

For the term I_9, let $\alpha_0\leq\alpha$ with $|\alpha_0|=1$, then

$$I_9=-\langle\partial_x^{\alpha-\alpha_0}G_2,\partial_x^{\alpha+\alpha_0}v\rangle\leq\epsilon\|\partial_x^{\alpha+\alpha_0}v\|^2+C_\epsilon\|\partial_x^{\alpha-\alpha_0}G_2\|^2. \tag{3.24}$$

Similar to the estimate of (3.21), we have

$$\|\partial_x^{\alpha-\alpha_0}G_2\|^2\leq C\|\tilde{U}\|_{N-1}^2\|\nabla\tilde{U}\|_N^2. \tag{3.25}$$

Thus, it follows from (3.24) and (3.25) that

$$I_9 \le \epsilon\|\partial_x^{\alpha+\alpha_0} v\|^2 + C\|\tilde{U}\|_{N-1}^2\|\nabla\tilde{U}\|_N^2. \tag{3.26}$$

Notice that (3.21) and (3.22) imply

$$\|\partial_x^\alpha G_1\|^2 \le C\left(\|\partial_x^\alpha(\nabla\tilde{\sigma}\cdot\tilde{v})\|^2 + \|\partial_x^\alpha(\tilde{\sigma}\nabla\cdot\tilde{v})\|^2\right) \le C\|\nabla\tilde{U}\|_{N-2}^2\|\nabla\tilde{U}\|_N^2. \tag{3.27}$$

Therefore, we derive from $(2.4)_1$, (3.27) and the Cauchy-Schwartz inequality that

$$\begin{aligned} I_{10} &= -\frac{\kappa'}{\gamma}\langle\partial_x^\alpha\Delta\sigma, -\partial_x^\alpha\sigma_t + \partial_x^\alpha G_1(\tilde{U})\rangle \\ &= -\frac{\kappa'}{\gamma}\langle\partial_x^\alpha\nabla\sigma, \partial_x^\alpha\nabla\sigma_t\rangle - \frac{\kappa'}{\gamma}\langle\partial_x^\alpha\Delta\sigma, \partial_x^\alpha G_1(\tilde{U})\rangle \\ &\le -\frac{\kappa'}{2\gamma}\frac{d}{dt}\|\partial_x^\alpha\nabla\sigma\|^2 + \epsilon\|\partial_x^\alpha\Delta\sigma\|^2 + C_\epsilon C\|\nabla\tilde{U}\|_{N-2}^2\|\nabla\tilde{U}\|_N^2. \end{aligned} \tag{3.28}$$

Moreover, it holds that

$$I_{11} = -\lambda_2\langle\partial_x^{\alpha+\alpha_0}v, \partial_x^{\alpha-\alpha_0}f\rangle \le \epsilon\|\partial_x^{\alpha+\alpha_0}v\|^2 + C_\epsilon C\|f\|_{N-1}^2. \tag{3.29}$$

where α_0 is defined in (3.24). Combining (3.14), (3.23), (3.26), (3.28) and (3.29), if ϵ is small enough, we have

$$\begin{aligned} &\frac{d}{dt}\left(\|\partial_x^\alpha\sigma\|_1^2 + \|\partial_x^\alpha v\|^2\right) + \|\partial_x^\alpha\nabla v\|^2 + \|\partial_x^\alpha\nabla\cdot v\|^2 \\ &\le \epsilon C\|\partial_x^\alpha\sigma\|^2 + \epsilon C\|\partial_x^\alpha\Delta\sigma\|^2 + C_\epsilon C\left(\|\nabla\tilde{U}\|_{N-2}^2\|\nabla\tilde{U}\|_N^2 + \|f\|_{N-1}^2\right), \end{aligned} \tag{3.30}$$

where C depends only on ρ_∞, μ, ν and κ.

Now we turn to estimate $\|\partial_x^\alpha\Delta\sigma\|^2$ for $1 \le |\alpha| \le N$. As we did for the first order derivative estimate, applying ∂_x^α to $(2.4)_2$ and then taking the L^2 inner product with $\partial_x^\alpha\nabla\sigma$ on the resultant equation, we get from integrating by parts that

$$\begin{aligned} &\kappa'\|\partial_x^\alpha\Delta\sigma\|^2 + \gamma\|\partial_x^\alpha\nabla\sigma\|^2 \\ &= -\langle\partial_x^\alpha v_t, \partial_x^\alpha\nabla\sigma\rangle + \mu'\langle\partial_x^\alpha\Delta v, \partial_x^\alpha\nabla\sigma\rangle + \nu'\langle\partial_x^\alpha\nabla(\nabla\cdot v), \partial_x^\alpha\nabla\sigma\rangle \\ &\quad + \langle\partial_x^\alpha G_2(\tilde{U}), \partial_x^\alpha\nabla\sigma\rangle + \lambda_2\langle\partial_x^\alpha f, \partial_x^\alpha\nabla\sigma\rangle \\ &= I_{12} + I_{13} + I_{14} + I_{15} + I_{16}. \end{aligned} \tag{3.31}$$

The first term I_{12} is controlled by

$$\begin{aligned} I_{12} &= -\frac{d}{dt}\langle\partial_x^\alpha v, \partial_x^\alpha\nabla\sigma\rangle + \langle\partial_x^\alpha v, \partial_x^\alpha\nabla\sigma_t\rangle \\ &= -\frac{d}{dt}\langle\partial_x^\alpha v, \partial_x^\alpha\nabla\sigma\rangle - \langle\partial_x^\alpha\nabla\cdot v, \partial_x^\alpha(-\gamma\nabla\cdot v + G_1(\tilde{U}))\rangle \\ &\le -\frac{d}{dt}\langle\partial_x^\alpha v, \partial_x^\alpha\nabla\sigma\rangle + 2\gamma\|\partial_x^\alpha\nabla\cdot v\|^2 + C\|\nabla\tilde{U}\|_{N-2}^2\|\nabla\tilde{U}\|_N^2. \end{aligned} \tag{3.32}$$

Here, in the last inequality of (3.32), we have used (3.27). By integrating by parts, the Cauchy-Schwartz inequality and (3.25), the other terms I_{13}-I_{15} can be estimated as follows.

$$I_{13} + I_{14} \le \frac{\kappa'}{4}\|\partial_x^\alpha\nabla^2\sigma\|^2 + C\left(\|\partial_x^\alpha\nabla v\|^2 + \|\partial_x^\alpha\nabla\cdot v\|^2\right), \tag{3.33}$$

$$I_{15} \leq \frac{\kappa'}{4}\|\partial_x^{\alpha+\alpha_0}\nabla\sigma\|^2 + C\|\tilde{U}\|_{N-1}^2\|\nabla\tilde{U}\|_N^2, \tag{3.34}$$

$$I_{16} \leq \frac{\kappa'}{4}\|\partial_x^{\alpha+\alpha_0}\nabla\sigma\|^2 + C\|f\|_{N-1}^2. \tag{3.35}$$

where α_0 is given in (3.24). Combining (3.31)-(3.35), we obtain

$$\begin{aligned}&\frac{d}{dt}\langle\partial_x^\alpha v, \partial_x^\alpha\nabla\sigma\rangle + \kappa'\|\partial_x^\alpha\Delta\sigma\|^2 + \gamma\|\partial_x^\alpha\nabla\sigma\|^2\\ &\leq C\left(\|\partial_x^\alpha\nabla v\|^2 + \|\partial_x^\alpha\nabla\cdot v\|^2\right) + C\left(\|\tilde{U}\|_{N-1}^2\|\nabla\tilde{U}\|_N^2 + \|f\|_{N-1}^2\right).\end{aligned} \tag{3.36}$$

Multiplying (3.36) with a suitably small constant $d_1 > 0$ and then adding the resultant equation to (3.30) gives

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_x^\alpha\sigma\|_1^2 + \|\partial_x^\alpha v\|^2 + d_1\langle\partial_x^\alpha v, \partial_x^\alpha\nabla\sigma\rangle\right) + \|\partial_x^\alpha\nabla\sigma\|_1^2 + \|\partial_x^\alpha\nabla v\|^2\\ &\leq \epsilon C\|\partial_x^\alpha\sigma\|^2 + CC_\epsilon\left(\|\nabla\tilde{U}\|_{N-2}^2\|\nabla\tilde{U}\|_N^2 + \|f\|_{N-1}^2\right),\end{aligned} \tag{3.37}$$

provided that d_1 and ϵ are small enough, where C depends only on ρ_∞, μ, ν and κ. Summing up α with $1 \leq |\alpha| \leq N$ in (3.37), then (3.13) follows immediately by the smallness of ϵ. This completes the proof of Lemma 3.4.

As a consequence of Lemmas 3.3-3.4, we have the following Corollary.

Corollary 3.1. *Let $n \geq 5$, $N \geq n+2$, then there exists two suitably small constants $d_0 > 0$ and $d_1 > 0$ such that*

$$\begin{aligned}&\frac{d}{dt}\left(\|\sigma(t)\|_{N+1}^2 + \|v(t)\|_N^2 + d_0\langle v, \nabla\sigma\rangle(t) + d_1\sum_{|\alpha|=1}^N\langle\partial_x^\alpha v, \partial_x^\alpha\nabla\sigma\rangle(t)\right) + \|\nabla\sigma(t)\|_{N+1}^2 + \|\nabla v(t)\|_N^2\\ &\leq C\left(\|\tilde{U}(t)\|_{N-1}^2\|\nabla\tilde{U}(t)\|_N^2 + \|f(t)\|_{H^{N-1}\cap L^1}^2\right),\end{aligned} \tag{3.38}$$

where C depends only on ρ_∞, μ, ν and κ.

Proof. Notice that, from the fact that $m-1 \geq 1$ and $m+1 \leq N-4$, we have

$$\|\nabla^3\sigma\|_{m-2}^2 + \|\nabla^2 v\|_{m-1}^2 \leq C\|\nabla^2\tilde{U}\|_{N-6}^2,$$

and

$$\|\tilde{U}\|_{m+1} \leq \|\tilde{U}\|_{N-4}^2.$$

Adding (3.37) to (3.1), we obtain (3.38) immediately by the smallness of ϵ. This completes the proof of Corollary 3.1.

4 Existence of time periodic solution

In this section, we will combine the linearized decay estimate Lemma 2.1 with the energy estimates Corollary 3.1 to show the existence of time periodic solution to (1.1). Now, we are ready to prove Theorem 1.1 as follows.

Proof of Theorem 1.1. The proof is divided into two steps.

Step 1. Suppose that there exists a time periodic solution $U^{per}(t) := (\sigma^{per}(x,t), v^{per}(x,t)), t \in \mathbb{R}$ of the system (2.1) with period T, and $U^{per}(t) \in X_{M_0}(0,T)$ for some constant $M_0 > 0$. Then it

solves (2.3) with initial date $U_s = U^{per}(s)$ for any given time $s \in \mathbb{R}$. Choosing $s = -kT$ for $k \in \mathbb{N}$. Clearly, $U^{per}(-kT) = U^{per}(0)$, thus (2.3) can be written in the mild form as

$$U^{per}(t) = \mathbb{S}(t, -kT)U^{per}(0) + \int_{-kT}^{t} \mathbb{S}(t,\tau)(G(U^{per})(\tau) + F(\tau))d\tau. \tag{4.1}$$

Denote $\mathbb{S}(t, -kT)U^{per}(0) := (\sigma_1^{per}(t), v_1^{per}(t))$. Applying Lemma 2.1 to $\mathbb{S}(t, -kT)U^{per}(0)$, we have

$$\begin{aligned}\|\sigma_1^{per}(t)\|_N &\le (1+t+kT)^{-\frac{n}{4}} \left(\|(\sigma_0^{per}, v_0^{per})\|_{L^1} + \|\sigma_0^{per}\|_N^2 + \|v_0^{per}\|_{N-1}^2\right) \\ &\longrightarrow 0 \quad as \quad k \to \infty.\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\|v_1^{per}(t)\|_{N-1} &\le (1+t+kT)^{-\frac{n}{4}} \left(\|(\sigma_0^{per}, v_0^{per})\|_{L^1} + \|\sigma_0^{per}\|_N^2 + \|v_0^{per}\|_{N-1}^2\right) \\ &\longrightarrow 0 \quad as \quad k \to \infty.\end{aligned} \tag{4.3}$$

Since $L^2 \cap L^1$ is dense in L^2, (4.2) and (4.3) still hold for $U^{per}(0) = (\sigma_0^{per}, v_0^{per}) \in H^N(\mathbb{R}^n) \times H^{N-1}(\mathbb{R}^n)$. On the other hand, denote

$$\mathbb{S}(t,\tau)(G(U^{per})(\tau) + F(\tau)) := (S_1(t,\tau), S_2(t,\tau)).$$

By using Lemma 2.1 again, we get

$$\|S_1(t,\tau)\|_N \le (1+t-\tau)^{-\frac{n}{4}} K_0, \quad \|S_2(t,\tau)\|_{N-1} \le (1+t-\tau)^{-\frac{n}{4}} K_0, \tag{4.4}$$

where

$$\begin{aligned}K_0 = &\|(G_1(U^{per}), G_2(U^{per}) + \lambda_2 f)(\tau)\|_{L^1} \\ &+ \|G_1(U^{per})(\tau)\|_N + \|(G_2(U^{per}) + \lambda_2 f)(\tau)\|_{N-1}.\end{aligned}$$

Then (4.4) guarantees the convergence of the integral in (4.1) since $\frac{n}{4} > 1$ when $n \ge 5$. Thus, letting $k \to \infty$ in (4.1), we obtain

$$U^{per}(t) = \int_{-\infty}^{t} \mathbb{S}(t,\tau)(G(U^{per}) + F)(\tau)d\tau. \tag{4.5}$$

For any $U = (\sigma, v) \in X_{M_0}(0,T)$, define

$$\Psi[U](t) = \int_{-\infty}^{t} \mathbb{S}(t,\tau)(G(U) + F)(\tau)d\tau.$$

Then (4.5) shows that U^{per} is a fixed point of $\Psi[U]$.

Conversely, suppose that Ψ has a unique fixed point, denoted by $U_1(t) = (\sigma_1, v_1)(t)$. We show that $U_1(t)$ is time periodic with period T. To this end, setting $U_2(t) = U_1(t+T)$. Since the period of f is T, the period of F is T too. Thus, we have

$$\begin{aligned}U_2(t) &= U_1(t+T) = \Psi[U_1](t+T) \\ &= \int_{-\infty}^{t+T} \mathbb{S}(t+T,\tau)(G(U_1)(\tau) + F(\tau))d\tau \\ &= \int_{-\infty}^{t} \mathbb{S}(t+T, s+T)\left(G(U_1)(s+T) + F(s+T)\right) ds \\ &= \int_{-\infty}^{t} \mathbb{S}(t,s)\left(G(U_2)(s) + F(s)\right) ds \\ &= \Psi[U_2](t)\end{aligned} \tag{4.6}$$

where we have used

$$\mathbb{S}(t+T, s+T) = \mathbb{S}(t, s).$$

Then by uniqueness, $U_2 = U_1$, which proves the periodicity of $U_1(t)$. Since $U_1(t)$ is differentiable with respect to t, it is the desired periodic solution of the system (2.1).

Step 2. Now, it remains to show that if (H1)-(H3) hold, and

$$\sup_{0\le t\le T} \|f(t)\|_{H^{N-1}\cap L^1}$$

is sufficiently small, then Ψ has a unique fixed point in the space $X_{M_0}(0,T)$ for some appropriate constant $M_0 > 0$. The proof is divided into two parts.

(i) Assume that $\tilde{U} = (\tilde{\sigma}, \tilde{v})$ in the system (2.4) is time periodic with period T. Denote $U = \Psi[\tilde{U}]$ with $U = (\sigma, v)$. Then by the same argument as (4.6), one can show that U is also time periodic with period T. Notice that U satisfies the system (2.4). Thus, for $n \ge 5$ and $N \ge n+2$, Corollary 3.1 holds. Integrating (3.38) in t over $[0,T]$ to get

$$\begin{aligned}
&\int_0^T \left(\|\nabla\sigma(t)\|_{N+1}^2 + \|\nabla v(t)\|_N^2\right) dt\\
&\le C\int_0^T \left(\|\tilde{U}(t)\|_{N-1}^2\|\nabla\tilde{U}(t)\|_N^2 + \|f(t)\|_{N-1}^2 + \|f(t)\|_{L^1}^2\right) dt\\
&\le C\sup_{0\le t\le T}\|\tilde{U}(t)\|_{N-1}^2\int_0^T \|\nabla\tilde{U}(t)\|_N^2 dt + \int_0^T \|f(t)\|_{H^{N-1}\cap L^1}^2 dt\\
&\le C|||\tilde{U}(t)|||^4 + CT\sup_{0\le t\le T}\|f(t)\|_{H^{N-1}\cap L^1}^2.
\end{aligned} \tag{4.7}$$

On the other hand, by Lemma 2.1, we have

$$\|\sigma(t)\|_N \le \int_{-\infty}^t (1+t-\tau)^{-\frac{n}{4}} K_1\, d\tau, \quad \|v(t)\|_{N-1} \le \int_{-\infty}^t (1+t-\tau)^{-\frac{n}{4}} K_1\, d\tau, \tag{4.8}$$

where

$$\begin{aligned}
K_1 = {} & \|(G_1(\tilde{U}), G_2(\tilde{U}) + \lambda_2 f)(\tau)\|_{L^1}\\
& + \|G_1(\tilde{U})(\tau)\|_N + \|(G_2(\tilde{U}) + \lambda_2 f)(\tau)\|_{N-1}.
\end{aligned} \tag{4.9}$$

From (2.2), (3.25) and (3.27), we easily deduce that

$$\begin{aligned}
\|(G_1(\tilde{U})(\tau)\|_{L^1} &\le C\|\nabla\tilde{U}(\tau)\|\|\tilde{U}(\tau)\|,\\
\|(G_1(\tilde{U})(\tau)\|_N &\le C\|\nabla\tilde{U}(\tau)\|_{N-2}\|\nabla\tilde{U}(\tau)\|_N,\\
\|G_2(\tilde{U}) + \lambda_2 f)(\tau)\|_{L^1} &\le C\|\nabla\tilde{U}(\tau)\|_1\|\tilde{U}(\tau)\| + C\|f(\tau)\|_{L^1},\\
\|G_2(\tilde{U}) + \lambda_2 f)(\tau)\|_{N-1} &\le C\|\tilde{U}(\tau)\|_{N-1}\|\nabla\tilde{U}(\tau)\|_N + C\|f(\tau)\|_{N-1}.
\end{aligned} \tag{4.10}$$

Combining (4.8)-(4.10), we obtain

$$\begin{aligned}
\|\sigma(t)\|_N &\le C\int_{-\infty}^t (1+t-\tau)^{-\frac{n}{4}} \left(\|\tilde{U}(\tau)\|_{N-1}\|\nabla\tilde{U}(\tau)\|_N + \|f(\tau)\|_{H^{N-1}\cap L^1}\right) d\tau\\
&\le C\sum_{j=0}^{\infty} A_j + C\int_{-\infty}^t (1+t-\tau)^{-\frac{n}{4}}\|f(\tau)\|_{H^{N-1}\cap L^1} d\tau\\
&\le C\sum_{j=0}^{\infty} A_j + C\sup_{0\le t\le T}\|f(t)\|_{H^{N-1}\cap L^1},
\end{aligned} \tag{4.11}$$

where

$$
\begin{aligned}
A_j &= C\int_{t-(j+1)T}^{t-jT}(1+t-\tau)^{-\frac{n}{4}}\|\tilde{U}(\tau)\|_{N-1}\|\nabla\tilde{U}(\tau)\|_N d\tau \\
&\le C\left(\int_{t-(j+1)T}^{t-jT}(1+t-\tau)^{-\frac{n}{2}}d\tau\right)^{\frac{1}{2}}\left(\int_{t-(j+1)T}^{t-jT}\|\tilde{U}(\tau)\|_{N-1}^2\|\nabla\tilde{U}(\tau)\|_N^2 d\tau\right)^{\frac{1}{2}} \\
&\le C(1+jT)^{-\frac{n}{4}}\sup_{0\le\tau\le T}\|\tilde{U}(\tau)\|_{N-1}\left(\int_0^T\|\nabla\tilde{U}(\tau)\|_N^2 d\tau\right)^{\frac{1}{2}} \\
&\le C(1+jT)^{-\frac{n}{4}}|||\tilde{U}|||^2
\end{aligned}
\tag{4.12}
$$

Since $\frac{n}{4}>1$ when $n\ge 5$, substituting (4.12) into (4.11) gives

$$\|\sigma(t)\|_N \le C|||\tilde{U}|||^2 + C\sup_{0\le t\le T}\|f(t)\|_{H^{N-1}\cap L^1}. \tag{4.13}$$

Similarly, it holds that

$$\|v(t)\|_{N-1} \le C|||\tilde{U}|||^2 + C\sup_{0\le t\le T}\|f(t)\|_{H^{N-1}\cap L^1}. \tag{4.14}$$

Thus, we deduce from (4.7), (4.13) and (4.14) that

$$|||\Psi[\tilde{U}]||| \le C_1|||\tilde{U}|||^2 + C_2\sup_{0\le t\le T}\|f(t)\|_{H^{N-1}\cap L^1}, \tag{4.15}$$

where C_1 and C_2 are some positive constants depending only on $\rho_\infty,\mu,\nu,\kappa$ and T.

(ii) Let $\tilde{U}_1=(\tilde{\sigma}_1,\tilde{v}_1)$ and $\tilde{U}_2=(\tilde{\sigma}_2,\tilde{v}_2)$ be time periodic functions with period T in the space $X_{M_0}(0,T)$, where $M_0>0$ will be determined below. Then similar to (i), we can get

$$|||\Psi[\tilde{U}_1]-\Psi[\tilde{U}_2]||| \le C_3\left(|||\tilde{U}_1|||+|||\tilde{U}_2|||\right)|||\tilde{U}_1-\tilde{U}_2|||, \tag{4.16}$$

where C_3 is a positive constant depending only on $\rho_\infty,\mu,\nu,\kappa$ and T. Choose $M_0>0$ and a sufficiently small constant $\delta>0$ such that

$$C_1M_0^2+C_2\delta\le M_0, \quad \text{and } 2C_3M_0<1 \tag{4.17}$$

That is,

$$\frac{1-\sqrt{1-4C_1C_2\delta}}{2C_1}\le M_0\le\min\left\{\frac{1+\sqrt{1-4C_1C_2\delta}}{2C_1},\frac{1}{2C_3},1\right\} \tag{4.18}$$

Notice that

$$\frac{1-\sqrt{1-4C_1C_2\delta}}{2C_1}\longrightarrow 0 \quad as \quad \delta\longrightarrow 0.$$

Then there exists a constant $\delta_0>0$ depending only on $\rho_\infty,\mu,\nu,\kappa$ and T such that if $0<\delta\le\delta_0$, the set of M_0 that satisfying (4.18) is not empty. For $0<\delta\le\delta_0$, when M_0 satisfies (4.18), Ψ is a contraction map in the complete space $X_{M_0}(0,T)$, thus Ψ has a unique fixed point in $X_{M_0}(0,T)$. This completes the proof of Theorem 1.1.

5 Stability of time periodic solution

This section is devoted to proving Theorem 1.2 on the stability of the obtained time periodic solution. We shall establish the global existence of smooth solutions to the Cauchy problem (1.1), (1.5).

First, let (ρ^{per}, u^{per}) be the time periodic solution obtained in Theorem 1.1 and (ρ, u) be the solution of the Cauchy problem (1.1), (1.5). Denote

$$(\sigma^{per}, v^{per}) = (\rho^{per} - \rho_\infty, \lambda_2 u^{per}),$$

$$(\sigma, v) = (\rho - \rho_\infty, \lambda_2 u).$$

Let $(\bar\sigma, \bar v) = (\sigma - \sigma^{per}, v - v^{per})$, then $(\bar\sigma, \bar v)$ satisfies

$$\begin{cases} \bar\sigma_t + \gamma\nabla\cdot\bar v = G_1(\bar\sigma + \sigma^{per}, \bar v + v^{per}) - G_1(\sigma^{per}, v^{per}), \\ \bar v_t - \mu'\Delta\bar v - \nu'\nabla(\nabla\cdot\bar v) + \gamma\nabla\bar\sigma - \kappa'\nabla\Delta\bar\sigma = G_2(\bar\sigma + \sigma^{per}, \bar v + v^{per}) - G_2(\sigma^{per}, v^{per}), \end{cases} \tag{5.1}$$

with the initial date

$$\bar\sigma|_{t=0} = \bar\sigma_0(x) = \rho_0(x) - \rho^{per}(0), \quad \bar v|_{t=0} = \bar v_0(x) = \lambda_2(u_0(x) - u^{per}(0)). \tag{5.2}$$

Define the solution space by $\bar X(0,\infty)$, where for $0 \le t_1 \le t_2 \le \infty$,

$$\bar X(t_1,t_2) = \left\{ (\bar\sigma,\bar v)(t,x) \,\middle|\, \begin{array}{c} \bar\sigma(t,x) \in C(t_1,t_2; H^{N-1}(\mathbb{R}^n)) \cap C^1(t_1,t_2; H^{N-3}(\mathbb{R}^n)), \\ \bar v(t,x) \in C(t_1,t_2; H^{N-2}(\mathbb{R}^n)) \cap C^1(t_1,t_2; H^{N-4}(\mathbb{R}^n)), \\ \nabla\bar\sigma(t,x) \in L^2(t_1,t_2; H^{N-1}(\mathbb{R}^n)), \nabla\bar v(t,x) \in L^2(t_1,t_2; H^{N-2}(\mathbb{R}^n)), \end{array} \right\} \tag{5.3}$$

with the norm

$$\|(\bar\sigma,\bar v)(t)\nparallel^2 := \sup_{t_1\le t\le t_2}\left\{\|\bar\sigma(t)\|_{N-1}^2 + \|\bar v(t)\|_{N-2}^2\right\} + \int_{t_1}^{t_2}\left(\|\nabla\bar\sigma(t)\|_{N-1}^2 + \|\nabla\bar v(t)\|_{N-2}^2\right)dt. \tag{5.4}$$

Notice that $(\sigma^{per}, v^{per}) \in \bar X(0,T)$.

By using the dual argument and iteration technique as [12], one can prove the following local existence of the Cauchy problem (5.1), (5.2). We omit the proof here for brevity.

Lemma 5.1. *(Local existence) Under the assumptions of Theorem 1.1, suppose that $(\bar\sigma_0, \bar v_0) \in H^{N-1}(\mathbb{R}^n) \times H^{N-2}(\mathbb{R}^n)$ and $\inf\rho_0(x) > 0$. Then there exists a positive constant T_0 depending only on $\|(\bar\sigma_0,\bar v_0)\nparallel$ such that the Cauchy problem (5.1), (5.2) admits a unique classical solution $(\bar\sigma,\bar v) \in \bar X(0,T_0)$ which satisfies*

$$\|(\bar\sigma,\bar v)(t)\nparallel \le C_4\|(\bar\sigma_0,\bar v_0)\nparallel,$$

where C_4 is a positive constant independent of $\|(\bar\sigma_0,\bar v_0)\nparallel$.

As usual, the global existence will be obtained by a combination of the local existence result Lemma 5.1 and the a priori estimate below.

Lemma 5.2. *(A priori estimate) Under the assumptions of Lemma 5.1, suppose that the Cauchy problem (5.1), (5.2) has a unique classical solution $(\bar\sigma,\bar v) \in \bar X(0,T_1)$ for some positive constant T_1. Then there exists two small constants $\delta > 0$ and $C_5 > 0$ which are independent of T_1 such that if*

$$\sup_{0\le t\le T_1}\|(\bar\sigma,\bar v)(t)\nparallel \le \delta, \tag{5.5}$$

it holds that

$$\|\bar{\sigma}(t)\|_{N-1}^2 + \|\bar{v}(t)\|_{N-2}^2 + \int_0^t \left(\|\nabla\bar{\sigma}(\tau)\|_{N-1}^2 + \|\nabla\bar{v}(\tau)\|_{N-2}^2\right) d\tau \le C_5 \left(\|\bar{\sigma}_0\|_{N-1}^2 + \|\bar{v}_0\|_{N-2}^2\right) \quad (5.6)$$

for all $t \in [0, T_1]$.

Proof. Noticing that some smallness conditions can be imposed on (σ^{per}, v^{per}), without loss of generality, we may assume $|||(\sigma^{per}, v^{per})||| \le \epsilon$ with $\epsilon > 0$ being sufficiently small. Then by the similar argument as in the proof of Lemmas 3.3-3.4, we can obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\bar{U}\|^2 + \|\nabla\bar{\sigma}\|^2 + d_2\langle\bar{v}, \nabla\bar{\sigma}\rangle\right) + \|\nabla\bar{v}\|^2 + \|\nabla\bar{\sigma}\|_1^2 \\ &\le \epsilon C\left(\|\nabla^3\sigma\|_{N-7}^2 + \|\nabla^2\bar{v}\|_{N-6}^2\right),\end{aligned} \quad (5.7)$$

and

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\bar{\sigma}\|_{N-2}^2 + \|\nabla\bar{v}\|_{N-3}^2 + d_3\sum_{|\alpha|=1}^{N-2}\langle\partial_x^\alpha\bar{v}, \partial_x^\alpha\nabla\bar{\sigma}\rangle\right) + \|\nabla^2\bar{\sigma}\|_{N-2}^2 + \|\nabla^2\bar{v}\|_{N-3}^2 \\ &\le \epsilon C\left(\|\nabla\bar{\sigma}\|^2 + \|\nabla\bar{v}\|^2\right),\end{aligned} \quad (5.8)$$

where $d_2 > 0$ and $d_3 > 0$ are some suitably small constants, and C is a constant depending only on ρ_∞, μ, ν and κ. Adding (5.8) to (5.7), it holds

$$\begin{aligned}&\frac{d}{dt}\left(\|\bar{\sigma}\|_{N-1}^2 + \|\bar{v}\|_{N-2}^2 + d_2\langle\bar{v}, \nabla\bar{\sigma}\rangle + d_3\sum_{|\alpha|=1}^{N-2}\langle\partial_x^\alpha\bar{v}, \partial_x^\alpha\nabla\bar{\sigma}\rangle\right) \\ &+\|\nabla\bar{\sigma}\|_{N-1}^2 + \|\nabla\bar{v}\|_{N-2}^2 \le 0,\end{aligned} \quad (5.9)$$

provided that ϵ is sufficiently small. Integrating (5.9) in t over $(0, t)$, one can immediately get (5.6) since

$$\|\bar{\sigma}\|_{N-1}^2 + \|\bar{v}\|_{N-2}^2 + d_2\langle\bar{v}, \nabla\bar{\sigma}\rangle + d_3\sum_{|\alpha|=1}^{N-2}\langle\partial_x^\alpha\bar{v}, \partial_x^\alpha\nabla\bar{\sigma}\rangle \sim \|\bar{\sigma}\|_{N-1}^2 + \|\nabla\bar{v}\|_{N-2}^2.$$

by the smallness of d_2 and d_3. This completes the proof of Lemma 5.2.

Proof of Theorem 1.2. By Lemmas 5.1-5.2 and the continuity argument, the Cauchy problem (5.1), (5.2) admits a unique solution $(\bar{\sigma}, \bar{v})$ globally in time, which satisfies (1.6) and (1.7). Then all the statements in Theorem 1.2 follow immediately. This completes the proof of Theorem 1.2.

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